

A CO-DIMENSION 3 SUB-RIEMANNIAN STRUCTURE ON GROMOLL-MEYER EXOTIC SPHERE

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ABSTRACT. We construct a co-dimension 3 completely non-holonomic sub-bundle on the Gromoll-Meyer exotic 7 sphere based on its realization as a base space of a $\mathrm{Sp}(2)$ -principal bundle with the structure group $\mathrm{Sp}(1)$. The same method is valid for constructing a co-dimension 3 completely non-holonomic sub-bundle on the standard 7 sphere (or more general on a $4n + 3$ dimensional standard sphere). In the latter case such a construction based on the Hopf bundle is well-known. Our method provides an alternated simple proof for the standard sphere \mathbb{S}^7 .

CONTENTS

1. Introduction	1
2. Principal bundles and horizontal subspaces	3
2.1. First step : Candidate of a non-holonomic sub-bundle	3
2.2. Second step : Bracket calculation	4
2.3. Third step : Total space P is a compact group	4
3. A co-dimension 3 sub-Riemannian structure on the standard 7 sphere	5
4. A co-dimension 3 sub-Riemannian structure on the Gromoll-Meyer exotic 7 sphere	8
5. Proof of the main Theorem	10
5.1. Lemmas from Gromoll-Meyer	10
5.2. Adjoint action and a characterization of the sub-bundle	11
5.3. Various linear bases of $\mathfrak{sp}(2)$	12
5.4. Final part of the proof	18
References	20

1. INTRODUCTION

In the area of differential geometry it is a natural problem to decide whether a “famous manifold” has a particular geometric structure. In this paper we deal with a so called sub-Riemannian structure and explicitly construct an example of co-dimension three on one of the exotic 7 spheres.

keywords: *Non-holonomic sub-bundle, contact structure, principal bundle, horizontal subspace*

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A sub-Riemannian structure on a manifold M is defined through a sub-bundle \mathcal{H} in the tangent bundle $T(M)$ equipped with an inner product and such that evaluations of vector fields taking values in \mathcal{H} together with their iterated Lie brackets span the whole tangent space at each point. Together with these data M is referred to as a sub-Riemannian manifold. A sub-bundle of the above type is called *completely non-holonomic* or *bracket generating* and - in a sense - this notion is opposite to a foliation structure. Corresponding to the *Frobenius theorem* global connectivity on connected components by (piecewise) horizontal curves is a basic geometric property of a sub-Riemannian structure (see [4]). From an analytic point of view the bracket generating property implies sub-ellipticity of corresponding second order differential operators ("*Sum of Squares*") [10]. On manifolds with this structure we can define an operator, called sub-Laplacian, which reflects various geometric properties (see for examples, [15, 13, 6, 7] and references therein). So one may say that the interest in the existence of a sub-Riemannian structure on a given manifold is caused by its various geometric and analytic implications.

Contact manifolds are among the most studied sub-Riemannian structures. Recall that a contact structure is of co-dimension one and can be described in terms of a special kind of one form. In most of the concrete cases, the completely non-holonomic sub-bundle admits a natural inner product which is the restriction of a given Riemannian metric. Some examples of this type originate from the theory of dynamical systems (see, for example [1]).

It was proved in [9] that every Brieskorn manifold has a contact structure and a generalization to a submanifold of a non-compact Kähler manifold has been given in [16]. Here the one form for defining a contact structure is given as a restriction of a one-form θ whose differential $d\theta$ is the Kähler form.

In particular, a 7-dimensional Brieskorn manifold, which is an exotic sphere and realized as a submanifold in the complex vector space \mathbb{C}^{10} , has a contact structure. However, it is not known whether a 7 dimensional exotic sphere has a higher co-dimensional sub-Riemannian structure. We mention that the standard 7 sphere has such a structure of co-dimension 3 (there are several). This has been proved in [12] (see [15], [13], and [2]).

In this paper we show that the Gromoll-Meyer exotic sphere has a co-dimension 3 sub-Riemannian structure. Our description is valid also for the standard case constructed in [12].

So, in §2 we define a candidate of a completely non-holonomic sub-bundle on the base space of a principal bundle and explain a method consisting of three steps for proving that it is completely non-holonomic.

Then in §3 we apply the method to the standard 7 sphere S^7 and give a simple proof of the existence of a co-dimension 3 completely non-holonomic sub-bundle recovering a result in [12]. In §4 and §5, we show our main theorem namely that the Gromoll-Meyer exotic 7 sphere Σ_{GM}^7 has a co-dimension 3 completely non-holonomic sub-bundle.

Parts of the arguments are based on methods in linear algebra. The non-trivial part consists in selecting four local vector fields according to the point among the candidates which generate (together with the evaluation of their brackets) the whole tangent space at each point in Σ_{GM}^7 .

2. PRINCIPAL BUNDLES AND HORIZONTAL SUBSPACES

We explain a standard procedure of defining a sub-bundle in the tangent bundle on the base space of a principal bundle by considering a possible extension of the structure group. By this method we obtain our "candidate" of a completely non-holonomic sub-bundle on the base space of the principal bundle.

2.1. First step : Candidate of a non-holonomic sub-bundle. Let $\pi_G : P \rightarrow N$ be a principal bundle with the structure group G . We denote the action of G on P by

$$E : P \times G \rightarrow N \cong P/G, \quad (p, g) \mapsto E_g(p) := p \cdot g^{-1}$$

and assume that the action is isometric with respect to a Riemannian metric

$$(\cdot, \cdot)_p : T_p(P) \times T_p(P) \rightarrow \mathbb{R}.$$

Let $K \subset G$ be a closed subgroup, then the restriction of the action E to the subgroup K gives a principal bundle

$$\pi_K : P \rightarrow P/K =: M$$

with the structure group K .

We have two orthogonal decomposition of the tangent bundle $T(P)$

$$T(P) = V^G \oplus H^G = V^K \oplus H^K, \quad V^K \subset V^G, \quad H^K \supset H^G,$$

where V^G is the sub-bundle of $T(P)$ tangent to the action of G and H^G is its orthogonal complement. V^K and H^K are defined in the same way.

Let \mathfrak{g} and \mathfrak{k} be the Lie algebra of the group G and K , respectively. Let $A \in \mathfrak{g}$ then we denote by \tilde{A} the vector field on P defined by

$$\tilde{A}(f)(p) = \tilde{A}_p(f) := \frac{d}{dt} (f(p \cdot e^{tA}))|_{t=0}.$$

Recall that \tilde{A} is called "fundamental vector field" corresponding to $A \in \mathfrak{g}$. It holds

$$(2.1) \quad (dE_g)_p(\tilde{A}_p) = \widetilde{Ad_g(A)}_{E_g(p)},$$

$$(2.2) \quad (dE_g)_p(H_p^G) = H_{E_g(p)}^G \text{ for } g \in G$$

and

$$(2.3) \quad (dE_k)_p(H_p^G) = H_{E_k(p)}^G \text{ for } k \in K.$$

Also we have

$$(2.4) \quad (dE_k)_p(H_p^K) = H_{E_k(p)}^K \text{ for } k \in K.$$

Then by the relation

$$(d\pi_G)_p(H_p^G) = (d\pi_G)_{E_g(p)}((dE_g)_p(H_p^G)) = (d\pi_G)_{E_g(p)}(H_{E_g(p)}^G)$$

and the assumption that E acts isometrically, we can define a Riemannian metric on $T(N)$, which is called a "submersion metric". The splitting $T(P) = V^G \oplus H^G$ gives a connection on the principal bundle

$$\pi_G : P \rightarrow N \cong P/G.$$

A similar result holds for the principal bundle $\pi_K : P \rightarrow M \cong P/K$. Moreover, the relation

$$(dE_k)_p(H_p^G) = H_{E_k(p)}^G$$

allows us to descend H^G to a sub-bundle in $T(P/K)$, which we denote by \mathcal{H} .

Here our aim is to show that the sub-bundle \mathcal{H} is a completely non-holonomic sub-bundle on $T(P/K)$ for particular cases.

2.2. Second step : Bracket calculation. Let X and Y be local vector fields around a point $q \in P/K \cong M$ taking values in \mathcal{H} and we denote by \tilde{X} and \tilde{Y} their horizontal lifts according to the connection defined by the horizontal sub-bundle H^K . Then we can assume that both lifts take values in H^G :

$$d\pi_K(\tilde{X}) = X, \quad d\pi_K(\tilde{Y}) = Y.$$

So instead of calculating the bracket $[X, Y]$, we calculate $[\tilde{X}, \tilde{Y}]$. Then we have

$$d\pi_K([\tilde{X}, \tilde{Y}]) = [d\pi_K(\tilde{X}), d\pi_K(\tilde{Y})] = [X, Y].$$

2.3. Third step : Total space P is a compact group. We assume that the total space P itself is a Lie group with the Lie algebra \mathfrak{p} and is equipped with an invariant metric (\cdot, \cdot) under the action of the structure group G .

We denote the left and right multiplication by

$$L_a : P \rightarrow P, \quad L_a(x) = a \cdot x \quad \text{and} \quad R_a : P \rightarrow P, \quad R_a(x) = x \cdot a, \quad (x, a \in P),$$

respectively. As usual the tangent space $T(P)$ is identified with $P \times \mathfrak{p}$ through the map

$$\mathfrak{p} \ni u \mapsto \tilde{u}_p \in T_p(P),$$

where \tilde{u} is a left invariant vector field with the value u at the identity, i.e.

$$\tilde{u}_p = (dL_p)_{Id}(u).$$

Equivalently we may define \tilde{u} in form of a derivation as:

$$\tilde{u}_p(f) = \tilde{u}(f)(p) = \frac{d}{dt} (f(p \cdot e^{tu}))|_{t=0}, \quad \text{where} \quad f \in C^\infty(P).$$

Definition 2.1. Let $A \in \mathfrak{g}$ and define a one form θ^A on P by

$$\theta_p^A : T_p(P) \rightarrow \mathbb{R}, \quad \theta_p^A(U) := (\tilde{A}_p, U)_p,$$

where \tilde{A} is the fundamental vector field corresponding to $A \in \mathfrak{g}$. So, the subspaces H_p^G and H_p^K are characterized by

$$\begin{aligned} H_p^G &= \{ U \in T_p(P) \mid \theta_p^A(U) = 0 \text{ for } \forall A \in \mathfrak{g} \}, \\ H_p^K &= \{ U \in T_p(P) \mid \theta_p^A(U) = 0 \text{ for } \forall A \in \mathfrak{k} \}. \end{aligned}$$

Let $p \in P$ and put $\mathfrak{h}_p = (dL_{p^{-1}})_p(H_p^G)$. In order to prove the complete non-holonomic property of the sub-bundle \mathcal{H} we show:

Proposition 2.2. *For each $q \in P/K$, there is $p \in P$ such that $\pi_K(p) = q$ and*

$$(2.5) \quad \{ \tilde{A}_{Id} \mid A \in \mathfrak{k} \} + \mathfrak{h}_p + [\mathfrak{h}_p, \mathfrak{h}_p] = \mathfrak{p}.$$

We show this property for the standard 7 sphere case. In case to the Gromoll-Meyer exotic sphere we prove

$$(2.6) \quad Ad_p(\{ \tilde{A}_{Id} \mid A \in \mathfrak{k} \} + \mathfrak{h}_p + [\mathfrak{h}_p, \mathfrak{h}_p]) = \mathfrak{p}.$$

Fix a point $p \in P$ and let \tilde{X}^i , $i = 0, 1, \dots$ be horizontal vector fields defined locally around p which take values in H^G and form a basis of H_p^G . Also let $v_i \in \mathfrak{h}_p$ such that $(dL_p)_{Id}(v_i) = \tilde{X}_p^i$, then $\{v_i\}$ is a basis of \mathfrak{h}_p and we show

Proposition 2.3.

$$[\tilde{X}^i, \tilde{X}^j]_p \pm (dL_p)_{Id}([v_i, v_j]) \in H_p^G,$$

for particular pairs of horizontal vector fields \tilde{X}^i or some type of sums of brackets of such vector fields

$$\sum c_{ij}^\ell [\tilde{X}^i, \tilde{X}^j]_p \pm (dL_p)_{Id} \left(\sum c_{ij}^\ell [v_i, v_j] \right) \in H_p^G.$$

The sign \pm will be chosen according to the cases.

Finally, we show that the vectors $\{\tilde{X}_p^i\}_i$ and $\{\sum c_{ij}^\ell [\tilde{X}^i, \tilde{X}^j]_p\}_\ell$ and evaluations $\{\tilde{A}_p \mid A \in \mathfrak{g}\}$ of fundamental vector fields span the tangent space $T_p(P)$.

3. A CO-DIMENSION 3 SUB-RIEMANNIAN STRUCTURE ON THE STANDARD 7 SPHERE

We give a simple proof for the existence of a co-dimension 3 completely non-holonomic sub-bundle on the standard 7 sphere (cf. [12] for a more direct approach). The method is valid for all $4n + 3$ dimensional standard spheres.

First, we describe a co-dimension 3 sub-bundle in the tangent bundle of the standard 7 sphere S^7 .

Let \mathbb{H} be the quaternion number field:

$$\mathbb{H} = \{h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} \mid h_i \in \mathbb{R}\},$$

with the usual product and conjugation operations:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \text{ etc.}$$

$$\bar{h} = h_0 - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k} \quad \text{and} \quad |h| = \sqrt{h\bar{h}}.$$

Let $\text{Sp}(2)$ be the group of quaternionic symplectic 2×2 matrices:

$$\text{Sp}(2) = \left\{ p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \mid x, y, w, z \in \mathbb{H}, p \cdot p^* = p^* \cdot p = Id \right\},$$

where

$$p^* = \begin{pmatrix} \bar{x} & \bar{w} \\ \bar{y} & \bar{z} \end{pmatrix}$$

is the adjoint matrix of p . We denote its Lie algebra by $\mathfrak{sp}(2)$:

$$\mathfrak{sp}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \gamma \end{pmatrix} \mid \alpha = -\bar{\alpha}, \gamma = -\bar{\gamma}, \beta \in \mathbb{H} \right\}.$$

Let $G = \text{Sp}(1) \times \text{Sp}(1) = \{(\lambda, \mu) \mid |\lambda| = |\mu| = 1\}$ and write \mathfrak{g} for its Lie algebra. We define the action of the group G on $\text{Sp}(2)$ by the right multiplication:

$$R : \text{Sp}(2) \times G \rightarrow \text{Sp}(2), \quad (p; \lambda, \mu) \mapsto R_{(\lambda, \mu)}(p)$$

$$R_{(\lambda, \mu)}(p) = \begin{pmatrix} x\bar{\lambda} & y\bar{\mu} \\ w\bar{\lambda} & z\bar{\mu} \end{pmatrix} = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \bar{\mu} \end{pmatrix},$$

$$R_{(\lambda_1, \mu_1)} \circ R_{(\lambda, \mu)} = R_{(\lambda_1 \cdot \lambda, \mu_1 \cdot \mu)}$$

and we as well consider its restriction to the subgroup $K = \text{Sp}(1) \times \{Id\} \subset G$.

Then we have two principal bundles. One is (cf. [8])

$$\pi_G : \text{Sp}(2) \longrightarrow S^4, \quad p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \mapsto (2y \cdot \bar{z}, |y|^2 - |z|^2) \in S^4,$$

with the base space S^4 and the structure group $G \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, and the other

$$\pi_K : \mathrm{Sp}(2) \longrightarrow S^7, \quad p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \mapsto (y, z) \in S^7.$$

with the base space being the standard 7 sphere S^7 and with the structure group

$$K = \mathrm{Sp}(1) \times \{Id\} \subset G.$$

Since K is a normal subgroup of G , we obtain a principal bundle called *Hopf bundle*,

$$\pi_{Hp} : S^7 \longrightarrow S^4,$$

with the structure group $G/K \cong \mathrm{Sp}(1)$. We will denote the structure group action by $\overline{R} : S^7 \times G/K \rightarrow S^7$.

We identify (trivialize) the tangent bundle $T(\mathrm{Sp}(2))$ through the left invariant vector fields:

$$(3.1) \quad \mathrm{Sp}(2) \times \mathfrak{sp}(2) \cong T(\mathrm{Sp}(2)),$$

where the identification is given by

$$\mathfrak{sp}(2) \ni u \longmapsto \tilde{u}_p \in T_p(\mathrm{Sp}(2))$$

\tilde{u} is the left invariant vector field.

Let $\langle \bullet, \bullet \rangle$ be the inner product on $\mathfrak{sp}(2)$ given by

$$\langle u, v \rangle \stackrel{Def}{=} \mathrm{Re}(\mathrm{Tr} u \cdot v^*) = \mathrm{Re}(x \cdot \bar{a} + y \cdot \bar{b} + w \cdot \bar{c} + z \cdot \bar{d}),$$

for $u = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ and $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Since

$$\langle \mathrm{Ad}_g(u), \mathrm{Ad}_g(v) \rangle = \langle u, v \rangle,$$

we can define a left and right invariant Riemannian metric (\bullet, \bullet) on $\mathrm{Sp}(2)$ through the above identification (3.1).

Let $V_p^G \subset T_p(\mathrm{Sp}(2))$ be the tangent space to the fibers of the principal bundle $\pi_G : \mathrm{Sp}(2) \rightarrow S^4$, that is, at $p \in \mathrm{Sp}(2)$

$$V_p^G = \left\{ X \in T_p(\mathrm{Sp}(2)) \mid X = (dL_p)_{Id} \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, x = -\bar{x}, z = -\bar{z} \in \mathbb{H} \right\}$$

and let us denote by H^G the orthogonal complement to V^G with respect to the Riemannian metric (\bullet, \bullet) :

$$H_p^G = \left\{ Y \in T_p(\mathrm{Sp}(2)) \mid \langle (dL_{p^{-1}})_p(Y), u \rangle = 0, \right. \\ \left. \text{for any } u = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, x = -\bar{x}, z = -\bar{z} \in \mathbb{H} \right\}.$$

So, $Y \in H_p^G$ is of the form

$$Y = (dL_p)_{Id} \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}, \quad a \in \mathbb{H}.$$

In particular, at the identity element $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(2)$ and via the identification $T_{Id}(\mathrm{Sp}(2)) \cong \mathfrak{sp}(2)$ we have the orthogonal decomposition:

$$\mathfrak{sp}(2) \ni u = \begin{pmatrix} x & y \\ -\bar{y} & z \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \oplus \begin{pmatrix} 0 & y \\ -\bar{y} & 0 \end{pmatrix} \in V_{Id}^G \oplus H_{Id}^G.$$

For $p \in \mathrm{Sp}(2)$, let V_p^K be the tangent space to the orbit of the action K through a point $p \in \mathrm{Sp}(2)$, then

$$V_p^K = \left\{ X = (dL_p)_{Id}(u) \mid u = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, -\bar{x} = x \in \mathbb{H} \right\}.$$

The orthogonal complement H_p^K of V_p^K is

$$H_p^K = \left\{ X = (dL_p)(u) \mid u = \begin{pmatrix} 0 & y \\ -\bar{y} & z \end{pmatrix}, -\bar{z} = z, y \in \mathbb{H} \right\}.$$

Since the decomposition $T(\mathrm{Sp}(2)) \cong H^G \oplus V^G$ is Ad -equivariant (see (2.1), (2.2) and (2.3)) under the orthogonal action of the group G and the decomposition $T(\mathrm{Sp}(2)) \cong H^K \oplus V^K$ is Ad -equivariant under the orthogonal action of the group K , each sub-bundle H^G and H^K defines a connection on the principal bundles

$$\begin{aligned} \pi_G : \mathrm{Sp}(2) &\longrightarrow S^4, \\ \pi_K : \mathrm{Sp}(2) &\longrightarrow S^7, \end{aligned}$$

respectively. Moreover, since the sub-bundle H^G is Ad -equivariant with respect to the structure group G it defines not only a sub-bundle in H^K but also induces a sub-bundle \mathcal{H}^S in $T(S^7)$.

The sub-bundle H^G defines a connection of the Hopf bundle $\pi_{Hp} : S^7 \rightarrow S^4 \cong P^1(\mathbb{H})$, that is the sub-bundle $d\pi_K(H^G) = \mathcal{H}^S$ satisfies

$$(d\bar{R}_{\bar{g}})_x(\mathcal{H}_x^S) = (\mathcal{H}_{\bar{R}_{\bar{g}}(x)}^S), \quad \bar{g} \in G/K, \quad x \in P/K.$$

Theorem 3.1. [12] *The sub-bundle \mathcal{H}^S is completely non-holonomic of step 2.*

Proof. We denote by $\Gamma(\mathcal{H}^S)$ the space of vector fields taking values in \mathcal{H}^S . Then we show that the evaluations of vectors fields in $\Gamma(\mathcal{H}^S) + [\Gamma(\mathcal{H}^S), \Gamma(\mathcal{H}^S)]$ span the whole tangent space at each point in S^7 .

Fix a point $q \in S^7$ and let X, Y be vector fields in $\Gamma(\mathcal{H}^S)$ defined around q . We denote their horizontal lifts by \tilde{X} and \tilde{Y} defined around $p \in \mathrm{Sp}(2)$ with $\pi_K(p) = q$. We may take vectors $u, v \in H_{Id}^G \subset \mathfrak{sp}(2)$ such that

$$\tilde{X}_p = (dL_p)_{Id}(u), \quad \tilde{Y}_p = (dL_p)_{Id}(v),$$

Since the fundamental vector field \tilde{A} ($A \in \mathfrak{g}$) is left invariant on $\mathrm{Sp}(2)$, we have

$$\theta^A(\tilde{u})(p) \equiv \langle A, u \rangle = \text{constant}.$$

Hence

$$\theta^A([\tilde{X}, \tilde{Y}](p)) = \theta^A((dL_p)_{Id}([u, v])) = \langle A, [u, v] \rangle.$$

In fact, this can be seen as follows:

$$\begin{aligned} d\theta^A(\tilde{u}_p, \tilde{v}_p) &= d\theta^A(\tilde{X}_p, \tilde{Y}_p) \\ &= \tilde{X}_p(\theta^A(\tilde{Y})) - \tilde{Y}_p(\theta^A(\tilde{X})) - \theta^A([\tilde{X}, \tilde{Y}])(p) \\ &= -\theta^A([\tilde{X}, \tilde{Y}])(p), \end{aligned}$$

because by the definition of \tilde{X} and \tilde{Y} , $\theta^A(\tilde{X}) = \theta^A(\tilde{Y}) \equiv 0$. On the other hand

$$\begin{aligned} d\theta^A(\tilde{u}_p, \tilde{v}_p) &= \tilde{u}_p(\theta^A(\tilde{v})) - \tilde{v}_p(\theta^A(\tilde{u})) - \theta^A([\tilde{u}, \tilde{v}])(p) \\ &= -\theta^A([\tilde{u}, \tilde{v}])(p) = -\langle A, [u, v] \rangle. \end{aligned}$$

This means that

$$[\tilde{X}, \tilde{Y}]_p - (dL_p)_{Id}([u, v]) \in H_p^G$$

(see Proposition 2.3). Therefore it is enough to show that

$$H_{Id}^G + [H_{Id}^G, H_{Id}^G]_{Id} = \mathfrak{sp}(2).$$

For this purpose, we take a basis of $H_{Id}^G = \mathfrak{h}_{Id}$

$$u_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} [u_0, u_1] &= \begin{pmatrix} 2\mathbf{i} & 0 \\ 0 & -2\mathbf{i} \end{pmatrix}, \quad [u_0, u_2] = \begin{pmatrix} 2\mathbf{j} & 0 \\ 0 & -2\mathbf{j} \end{pmatrix}, \quad [u_0, u_3] = \begin{pmatrix} 2\mathbf{k} & 0 \\ 0 & -2\mathbf{k} \end{pmatrix}, \\ [u_1, u_2] &= \begin{pmatrix} 2\mathbf{k} & 0 \\ 0 & 2\mathbf{k} \end{pmatrix}, \quad [u_1, u_3] = -\begin{pmatrix} 2\mathbf{j} & 0 \\ 0 & 2\mathbf{j} \end{pmatrix}, \quad [u_2, u_3] = \begin{pmatrix} 2\mathbf{i} & 0 \\ 0 & 2\mathbf{i} \end{pmatrix}. \end{aligned}$$

Hence these 10 vectors span the tangent space $\mathfrak{sp}(2) \cong T_{Id}(\mathrm{Sp}(2))$, which shows Theorem 3.1 (see Propositions 2.2 and 2.3). \square

4. A CO-DIMENSION 3 SUB-RIEMANNIAN STRUCTURE ON THE GROMOLL-MEYER EXOTIC 7 SPHERE

First we recall the definition of an exotic 7 sphere (called Gromoll-Meyer exotic sphere) following the description in [8]. We define a co-dimension 3 sub-bundle in the tangent bundle of the Gromoll-Meyer exotic 7 sphere. In the next section it will be shown that this sub-bundle is 2 step completely non-holonomic.

Consider an action $E : \mathrm{Sp}(2) \times G \rightarrow \mathrm{Sp}(2)$ on $\mathrm{Sp}(2)$ where $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$,

$$\begin{aligned} E : \mathrm{Sp}(2) \times G &\rightarrow \mathrm{Sp}(2) \\ E_{(\lambda, \mu)}(p) &= \begin{pmatrix} \lambda x \bar{\mu} & \lambda y \\ \lambda w \bar{\mu} & \lambda z \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix} \begin{pmatrix} \bar{\mu} & 0 \\ 0 & 1 \end{pmatrix} \\ E_{(\lambda_1, \mu_1)} \circ E_{(\lambda, \mu)} &= E_{(\lambda_1 \cdot \lambda, \mu_1 \cdot \mu)} \end{aligned}$$

and its restriction E^Δ to the subgroup

$$\Delta = \{(\lambda, \lambda) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)\} \cong \mathrm{Sp}(1).$$

The total space $\mathrm{Sp}(2)$ is equipped with the left and right invariant metric as in §3. Then we have a principal bundle $P_{GM} = \{\mathrm{Sp}(2), \Delta, \Sigma_{GM}^7\}$ with the orthogonal action of the structure group Δ :

$$\pi_{GM} : \mathrm{Sp}(2) \longrightarrow \mathrm{Sp}(2)/\Delta := \Sigma_{GM}^7,$$

The base space is called *Gromoll-Meyer exotic 7 sphere* ([8]). Also we have a principal bundle $P_G = \{\mathrm{Sp}(2), G, S^4\}$ with the orthogonal action of the structure group G

$$\rho : \mathrm{Sp}(2) \longrightarrow \mathrm{Sp}(2)/G \cong S^4,$$

where the identification of the base space with S^4 is induced through the map

$$\begin{aligned} \mathrm{Sp}(2) &\longrightarrow S^4 \subset \mathbb{H} \times \mathbb{R}, \\ p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} &\longmapsto (2\overline{y} \cdot z, |y|^2 - |z|^2). \end{aligned}$$

For any $\lambda \in \mathfrak{sp}(1)$, we write $\lambda^+ = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$.

By definition of the action E , the fundamental vector field \tilde{A} for $A = (a, b) \in \mathfrak{sp}(1) \times \mathfrak{sp}(1) = \mathfrak{g}$ is given by

$$(4.1) \quad \tilde{A}_p = (dR_p)_{Id}(a \cdot Id) - (dL_p)_{Id}(b^+).$$

Hence for $u \in \mathfrak{sp}(2)$

$$\begin{aligned} \theta^A(\tilde{u})(p) &= (\tilde{A}_p, \tilde{u}_p)_p = ((dR_p)_{Id}(a \cdot Id) - (dL_p)_{Id}(b^+), \tilde{u}_p)_p \\ &= ((dR_p)_{Id}(a) - (dL_p)_{Id}(b^+), (dL_p)_{Id}(u))_p \\ &= \langle a \cdot Id, Ad_p(u) \rangle - \langle b^+, u \rangle. \end{aligned}$$

We consider two orthogonal decompositions of $T(\mathrm{Sp}(2))$ as in §3 by vertical and horizontal sub-bundles according to the principal bundles P_{GM} and P_G :

$$\begin{aligned} T(\mathrm{Sp}(2)) &= V^\Delta \oplus H^\Delta, \\ V_p^\Delta &= \left\{ (dR_p)_{Id}(\lambda \cdot Id) - (dL_p)_{Id}(\lambda^+) \mid \bar{\lambda} = -\lambda \right\}, \\ H_p^\Delta &= (V_p^\Delta)^\perp, \\ T(\mathrm{Sp}(2)) &= V^G \oplus H^G, \\ V_p^G &= \left\{ (dR_p)_{Id}(\lambda \cdot Id) - (dL_p)_{Id}(\mu^+) \mid \lambda = -\bar{\lambda}, \mu = -\bar{\mu} \right\}, \\ H_p^G &= (V_p^G)^\perp \\ &= (dL_p)_{Id} \left(\left\{ u \in \mathfrak{sp}(2) \mid u = \begin{pmatrix} 0 & \beta \\ -\beta & \gamma \end{pmatrix}, \text{ and } \mathrm{Tr}(Ad_p(u)) = 0 \right\} \right) \\ &=: (dL_p)_{Id}(\mathfrak{h}_p). \end{aligned}$$

Then

$$V^\Delta \subset V^G \quad \text{and} \quad H^\Delta \supset H^G$$

and both of H^Δ and H^G are “*Ad-equivariant*” with respect to the action of the structure group Δ and G , respectively. Hence they define a connection on each principal bundle P_{GM} and P_S . Since $H^G \subset H^\Delta$ and H^G is Ad-equivariant under the structure group action of Δ , H^G defines a sub-bundle \mathcal{H}^Σ of the tangent bundle $T(\Sigma_{GM}^7)$.

Now we state our main theorem.

Theorem 4.1. \mathcal{H}^Σ is a co-dimension 3 completely non-holonomic sub-bundle in $T(\Sigma_{GM}^7)$ and of step 2.

The proof of Theorem 4.1 will be given in the following section.

5. PROOF OF THE MAIN THEOREM

Let $q \in \Sigma_{GM}^7$ and $p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \pi_{GM}^{-1}(q) \subset \mathrm{Sp}(2)$. The group action of Δ allows us to choose p with the property that $xw^{-1} =: v$, ($w \neq 0$) such that v is of the form $v = v_0 + v_1 \mathbf{i}$. Then p is rewritten as

$$p = \begin{pmatrix} vw & y \\ w & -\bar{v}y \end{pmatrix} \in \mathrm{Sp}(2) \quad \text{and} \quad |w| = |y| = \frac{1}{\sqrt{|v|^2 + 1}}.$$

Let $\rho \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and take $A = (\rho, \rho) \in \mathfrak{sp}(1) \times \mathfrak{sp}(1)$. As before we denote by \tilde{A} the fundamental vector field of A . According to (4.1) we obtain:

$$(5.1) \quad \ell_\rho := \mathrm{Ad}_p(\tilde{A}_{Id}) = \begin{pmatrix} \rho - x\rho\bar{x} & -x\rho\bar{w} \\ -w\rho\bar{x} & \rho - w\rho\bar{w} \end{pmatrix}.$$

We divide the proof into 4 cases according to the possible values of v :

Case (I) : $x \neq 0$ and $w \neq 0$. We divide into three sub-cases,

Case (I-a) : $v_1 \neq 0$ and $v^2 \neq -1$,

Case (I-b) : $v^2 = -1$,

Case (I-r) : v is real,

Case (II) : $x = 0$ or $w = 0$.

The reason for dividing into these 4 cases will be apparent from formula (5.17) below. The cases (I-r) and (II) can be treated in a way similar to the case of the standard 7 sphere. We remark that case (I-a) is generic, that is the points in Σ_{GM}^7 having a fiber with such a property is open dense. So we give the proof for this case in full details.

5.1. Lemmas from Gromoll-Meyer. We recall some basic properties from [8]. Let $q \in \Sigma_{GM}^7$ and let X, Y be two local vector fields on Σ_{GM}^7 around a point q taking values in \mathcal{H}^Σ . By \tilde{X} and \tilde{Y} we denote their horizontal lifts to $P = \mathrm{Sp}(2)$. Then we may assume that $\tilde{X}, \tilde{Y} \in \Gamma(H^G)$, i.e.,

$$\theta^{\lambda \cdot Id}(\tilde{X}) = 0, \quad \theta^{\mu^+}(\tilde{X}) = 0, \quad \forall \lambda, \mu \in \mathfrak{sp}(1),$$

and the same for \tilde{Y} .

We fix a point $p \in \mathrm{Sp}(2)$, and put $(dL_p)_{Id}(u) = \tilde{X}_p$ and $(dL_p)_{Id}(v) = \tilde{Y}_p$ with $u, v \in \mathfrak{sp}(2)$ that is, $\tilde{u}_p = \tilde{X}_p$ and $\tilde{v}_p = \tilde{Y}_p$. Then

$$\begin{aligned} d\theta^{\lambda \cdot Id}(\tilde{X}_p, \tilde{Y}_p) &= \tilde{X}_p(\theta^{\lambda \cdot Id}(\tilde{Y})) - \tilde{Y}_p(\theta^{\lambda \cdot Id}(\tilde{X})) - \theta^{\lambda \cdot Id}([\tilde{X}, \tilde{Y}])(p) \\ &= -\theta^{\lambda \cdot Id}([\tilde{X}, \tilde{Y}])(p), \end{aligned}$$

since $\theta^{\lambda \cdot Id}(\tilde{Y}) \equiv 0$ and $\theta^{\lambda \cdot Id}(\tilde{X}) \equiv 0$. On the other hand

$$\begin{aligned} d\theta^{\lambda \cdot Id}(\tilde{X}_p, \tilde{Y}_p) &= d\theta^{\lambda \cdot Id}(\tilde{u}_p, \tilde{v}_p) \\ &= \tilde{u}_p(\theta^{\lambda \cdot Id}(\tilde{v})) - \tilde{v}_p(\theta^{\lambda \cdot Id}(\tilde{u})) - \theta^{\lambda \cdot Id}([\tilde{u}, \tilde{v}])(p) \\ &= \tilde{u}_p(\langle \lambda \cdot Id, \mathrm{Ad}_*(v) \rangle) - \tilde{v}_p(\langle \lambda \cdot Id, \mathrm{Ad}_*(u) \rangle) - \theta^{\lambda \cdot Id}(\mathrm{Ad}_p([u, v])) \\ &= \langle \lambda \cdot Id, \mathrm{Ad}_p([u, v]) \rangle - \langle \lambda \cdot Id, \mathrm{Ad}_p([v, u]) \rangle - \langle \lambda \cdot Id, \mathrm{Ad}_p([u, v]) \rangle \\ &= \langle \lambda \cdot Id, \mathrm{Ad}_p([u, v]) \rangle. \end{aligned}$$

Hence, if we put $(dL_p)_{Id}(Z) = [\tilde{X}, \tilde{Y}]_p$, then

Lemma 5.1. ([8]) For all $\lambda = -\bar{\lambda} \in \mathbb{H}$:

$$\theta^{\lambda \cdot Id}([\tilde{X}, \tilde{Y}](p) + \theta^{\lambda \cdot Id}([\tilde{u}, \tilde{v}](p)) = \langle \lambda \cdot Id, Ad_p(Z + [u, v]) \rangle = 0.$$

That is, let X and Y be any local vector fields on Σ_{GM}^7 taking values in \mathcal{H}^Σ with horizontal lifts \tilde{X} and \tilde{Y} around $p \in \text{Sp}(2)$. We may find u and v in $\mathfrak{sp}(2)$ such that

$$(dL_p)_{Id}(u) = \tilde{u}_p = \tilde{X}_p$$

and $(dL_p)_{Id}(v) = \tilde{v}_p = \tilde{Y}_p$. Let $Z \in \mathfrak{sp}(2)$ such that $(dL_p)_{Id}(Z) = [\tilde{X}, \tilde{Y}]_p$. Then above calculations says

$$(5.2) \quad \text{Tr}(Ad_p(Z + [u, v])) = 0.$$

Likewise, with the same notations as in the previous lemma we have:

Lemma 5.2. ([8]) For all $\lambda = -\bar{\lambda} \in \mathbb{H}$:

$$(5.3) \quad \theta^{\lambda^+}([\tilde{X}, \tilde{Y}](p) - \theta^{\lambda^+}([\tilde{u}, \tilde{v}](p)) = \langle \lambda^+, Z - [u, v] \rangle = 0.$$

5.2. Adjoint action and a characterization of the sub-bundle. We fix a point $p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \text{Sp}(2)$ and put

$$\mathfrak{h}_p := (dL_{p^{-1}})_p(H_p^G).$$

Then

$$\mathfrak{h}_p = \left\{ u = \begin{pmatrix} 0 & \beta \\ -\bar{\beta} & \gamma \end{pmatrix} \mid x\beta\bar{y} - y\bar{\beta}x + w\beta\bar{z} - z\bar{\beta}w + y\gamma\bar{y} + z\gamma\bar{z} = 0 \right\}.$$

Conversely, let

$$u = \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} \in \mathfrak{sp}(2)$$

and $Ad_{p^{-1}}(u)$ be of the form $\begin{pmatrix} 0 & \beta \\ -\bar{\beta} & \gamma \end{pmatrix}$ then

$$(5.4) \quad \bar{x}ax - \bar{w}bx + \bar{x}bw - \bar{w}aw = 0.$$

Hence

$$(5.5) \quad Ad_p(\mathfrak{h}_p) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} \mid \bar{x}ax - \bar{w}bx + \bar{x}bw - \bar{w}aw = 0 \right\}.$$

Let $p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \text{Sp}(2)$. We solve the equation (5.4) by considering the above two cases separately and fix a basis of the space $Ad_p(\mathfrak{h}_p)$:

Case (I) : Assume that $x \cdot w \neq 0$. Equation (5.4) is rewritten as

$$(5.6) \quad \bar{v} \cdot a \cdot v - a = \bar{b} \cdot v - \bar{v} \cdot b,$$

where $v = xw^{-1}$ and we have assumed that $v = v_0 + v_1\mathbf{i} \neq 0$. We present the solutions of the equation (5.6).

Put $\theta_a := \bar{v} \cdot a \cdot v - a$. Then the solutions are given as follows:

(S₁) Clearly, the pair $(a, b) = (0, v)$ is a solution.

(S₂) Define $b_a = -\frac{v\theta_a}{2|v|^2} = \frac{v \cdot a - |v|^2 a \cdot v}{2|v|^2}$, then

$$\bar{b}_a \cdot v - \bar{v} \cdot b_a = \frac{\theta_a \cdot |v|^2}{2|v|^2} + \frac{|v|^2 \cdot \theta_a}{2|v|^2} = \theta_a, \quad (a = -\bar{a} \in \mathbb{H}).$$

Hence the pair $(a, b) = (a, b_a)$ is a solution for any $a = -\bar{a} \in \mathbb{H}$.

With the solutions in (S₁) and (S₂) we define a basis u_0, u_i, u_j, u_k of $Ad_p(\mathfrak{h}_p)$ as follows:

$$(5.7) \quad u_0(v) = u_0 = \begin{pmatrix} 0 & v \\ -\bar{v} & 0 \end{pmatrix},$$

$$(5.8) \quad u_i(v) = u_i = \begin{pmatrix} \mathbf{i} & b_i \\ -\bar{b}_i & -\mathbf{i} \end{pmatrix} = \begin{pmatrix} 1 & \frac{v(1-|v|^2)}{2|v|^2} \\ \frac{\bar{v}(1-|v|^2)}{2|v|^2} & -1 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix},$$

$$(5.9) \quad u_j(v) = u_j = \begin{pmatrix} \mathbf{j} & b_j \\ -\bar{b}_j & -\mathbf{j} \end{pmatrix} = \begin{pmatrix} 1 & \frac{v-|v|^2\bar{v}}{2|v|^2} \\ \frac{\bar{v}-|v|^2v}{2|v|^2} & -1 \end{pmatrix} \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} := S(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix},$$

$$(5.10) \quad u_k(v) = u_k = \begin{pmatrix} \mathbf{k} & b_k \\ -\bar{b}_k & -\mathbf{k} \end{pmatrix} = S(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}.$$

Remark 5.3. By the definition of the space $Ad_p(\mathfrak{h}_p)$ these four matrices have the property that $\text{Tr}(u_0) = \text{Tr}(u_i) = \text{Tr}(u_j) = \text{Tr}(u_k) = 0$, and the $(1, 1)$ components of $Ad_{p^{-1}}(u_\rho)$ vanish for $\rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Next we explicitly calculate the commutators $[u_0, u_i], [u_0, u_j], [u_0, u_k], [u_i, u_j]$ and $[u_i, u_k]$, respectively.

(5.11)

$$\begin{aligned} [u_0, u_i] &= \begin{pmatrix} 1 - |v|^2 & -2v \\ -2\bar{v} & |v|^2 - 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \\ [u_0, u_j] &= \begin{pmatrix} \frac{v^2(1-\bar{v}^2)}{|v|^2} & -2v_0 \\ -2v_0 & \bar{v}^2 - 1 \end{pmatrix} \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} =: M(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}, \\ [u_0, u_k] &= \begin{pmatrix} \frac{v^2(1-\bar{v}^2)}{|v|^2} & -2v_0 \\ -2v_0 & \bar{v}^2 - 1 \end{pmatrix} \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix} = M(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}, \\ [u_i, u_j] &= \begin{pmatrix} 2 + \frac{v^2(1-|v|^2)(1-\bar{v}^2)}{2|v|^4} & -\frac{v_1(1-|v|^2)\mathbf{i}}{|v|^2} \\ -\frac{v_1(1-|v|^2)\mathbf{i}}{|v|^2} & 2 + \frac{(1-|v|^2)(1-\bar{v}^2)}{2|v|^2} \end{pmatrix} \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix} =: B(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}, \\ [u_i, u_k] &= -\begin{pmatrix} 2 + \frac{v^2(1-|v|^2)(1-\bar{v}^2)}{2|v|^4} & -\frac{v_1(1-|v|^2)\mathbf{i}}{|v|^2} \\ -\frac{v_1(1-|v|^2)\mathbf{i}}{|v|^2} & 2 + \frac{(1-|v|^2)(1-\bar{v}^2)}{2|v|^2} \end{pmatrix} \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} = -B(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}. \end{aligned}$$

5.3. Various linear bases of $\mathfrak{sp}(2)$. According to the 4 cases explained above we choose a basis of $Ad_p(\mathfrak{h}_p) + [Ad_p(\mathfrak{h}_p), Ad_p(\mathfrak{h}_p)]$ with the property that

$$\mathfrak{sp}(2) = Ad_p((dL_{p^{-1}})_p V_p^\Delta + \mathfrak{h}_p) + [Ad_p(\mathfrak{h}_p), Ad_p(\mathfrak{h}_p)].$$

(I-a) We assume $v_1 \neq 0, v^2 \neq -1$. Hence $|v|^2 - v^2 \neq 0$. Put

$$\alpha(v) = \frac{8|v|^4 + (1 - \bar{v}^2)(1 - |v|^2)(v^2 + |v|^2)}{2|v|^2(1 - \bar{v}^2)(|v|^2 - v^2)}$$

$$(5.12) \quad = \frac{4\bar{v}}{(1 - \bar{v}^2)(\bar{v} - v)} + \frac{(1 - |v|^2)(v + \bar{v})}{2|v|^2(\bar{v} - v)},$$

and define matrices $U_{\mathbf{j}}$ and $U_{\mathbf{k}}$ by

$$(5.13) \quad U_{\mathbf{j}}(v) = U_{\mathbf{j}} = \alpha(v) \cdot [u_0(v), u_{\mathbf{j}}(v)] - [u_{\mathbf{i}}(v), u_{\mathbf{k}}(v)] = T(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} \quad \text{and}$$

$$U_{\mathbf{k}}(v) = U_{\mathbf{k}} = \alpha(v) \cdot [u_0(v), u_{\mathbf{k}}(v)] + [u_{\mathbf{i}}(v), u_{\mathbf{j}}(v)] = T(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}.$$

The above definition of $\alpha(v)$ ensures that the traces of these two matrices vanish:

$$(5.14) \quad \text{Tr}(U_{\mathbf{j}}) = 0, \quad \text{and} \quad \text{Tr}(U_{\mathbf{k}}) = 0.$$

Proposition 5.4. *Under the assumptions that $xw \neq 0$, $(xw^{-1})^2 \neq -1$ and with our notation in (5.1) the 10 matrices*

$$\{u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, [u_0, u_{\mathbf{i}}], U_{\mathbf{j}}, U_{\mathbf{k}}, \ell_{\mathbf{i}}, \ell_{\mathbf{j}}, \ell_{\mathbf{k}}\}$$

form a linear basis (over \mathbb{R}) of the space $\mathfrak{sp}(2)$.

Proof. Assume that

$$\lambda_1 \ell_{\mathbf{i}} + \lambda_2 \ell_{\mathbf{j}} + \lambda_3 \ell_{\mathbf{k}} + c_0 u_0 + c_1 u_{\mathbf{i}} + c_2 u_{\mathbf{j}} + c_3 u_{\mathbf{k}} + d_1 [u_0, u_{\mathbf{i}}] + d_2 U_{\mathbf{j}} + d_3 U_{\mathbf{k}} = 0.$$

We show that the coefficients λ_j, c_j, d_j vanish. According to (5.14) we know that

$$\text{Tr } U_{\mathbf{j}} = \text{Tr } U_{\mathbf{k}} = 0,$$

and by the explicit expressions of the matrices $u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, [u_0, u_{\mathbf{i}}]$ their traces vanish too. Therefore we have:

$$\begin{aligned} & \text{Tr}(\lambda_1 \ell_{\mathbf{i}} + \lambda_2 \ell_{\mathbf{j}} + \lambda_3 \ell_{\mathbf{k}} + c_0 u_0 + c_1 u_{\mathbf{i}} + c_2 u_{\mathbf{j}} + c_3 u_{\mathbf{k}} + d_1 [u_0, u_{\mathbf{i}}] + d_2 U_{\mathbf{j}} + d_3 U_{\mathbf{k}}) \\ &= \text{Tr}(\lambda_1 \ell_{\mathbf{i}} + \lambda_2 \ell_{\mathbf{j}} + \lambda_3 \ell_{\mathbf{k}}) = 0. \end{aligned}$$

Hence

$$2(\lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k}) = x \cdot (\lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k}) \cdot \bar{x} + w \cdot (\lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k}) \cdot \bar{w}$$

and consequently,

$$2|\lambda| \leq |\lambda||x|^2 + |\lambda||w|^2 = |\lambda|,$$

which implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Since $v = v_0 + v_1 \mathbf{i}$ and the constant $\alpha(v) = \alpha_0(v) + \alpha_1(v) \mathbf{i}$ is of the same form, the equation

$$c_0 u_0 + c_1 u_{\mathbf{i}} + c_2 u_{\mathbf{j}} + c_3 u_{\mathbf{k}} + d_1 [u_0, u_{\mathbf{i}}] + d_2 U_{\mathbf{j}} + d_3 U_{\mathbf{k}} = 0$$

can be separated into a system of two

$$(5.15) \quad c_0 u_0 + c_1 u_{\mathbf{i}} + d_1 [u_0, u_{\mathbf{i}}] = 0,$$

$$(5.16) \quad c_2 u_{\mathbf{j}} + c_3 u_{\mathbf{k}} + d_2 U_{\mathbf{j}} + d_3 U_{\mathbf{k}} = 0.$$

The equation (5.15) is rewritten as

$$\begin{aligned} c_1 + d_1(1 - |v|^2) &= 0, \\ -c_0 v \mathbf{i} + c_1 \frac{v(1 - |v|^2)}{2|v|^2} - 2d_1 v &= 0, \\ c_0 \bar{v} \mathbf{i} + c_1 \frac{\bar{v}(1 - |v|^2)}{2|v|^2} - 2d_1 \bar{v} &= 0. \end{aligned}$$

Hence we have

$$c_0 = 0, \quad c_1 = 0 \text{ and } d_1 = 0.$$

Equation (5.16) is equivalent to the system

$$0 = c_2 S(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} + c_3 S(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix} + d_2 T(v) \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} + d_3 T(v) \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix},$$

where

$$S(v) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{v-|v|^2\bar{v}}{2|v|^2} \\ \frac{v-|v|^2\bar{v}}{2|v|^2} & -1 \end{pmatrix}, \text{ and we put } T(v) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.$$

Multiplication from the right by $\mathbf{j} \cdot \text{Id}$ implies that:

$$0 = (c_2 + c_3 \mathbf{i}) S(v) + (d_2 + d_3 \mathbf{i}) T(v) = 0.$$

We need to show that the two complex matrices $S(v)$ and $T(v)$ are linearly independent over the complex numbers \mathbb{C} . Here the components t_{11} and t_{12} are explicitly given as

$$\begin{aligned} t_{11} &= \alpha(v) \cdot \frac{v^2(1-\bar{v}^2)}{|v|^2} + 2 + \frac{v^2(1-|v|^2)(1-\bar{v}^2)}{2|v|^4} = \frac{(1+|v|^2) \cdot v \cdot (1+\bar{v}^2)}{|v|^2(\bar{v}-v)}, \\ t_{12} &= -\alpha(v) \cdot (v+\bar{v}) - \frac{(1-|v|^2)(v-\bar{v})}{2|v|^2} = 2 \frac{(1+|v|^2)(1+\bar{v}^2)}{(1-\bar{v}^2)(v-\bar{v})}. \end{aligned}$$

These formulas are obtained by using the expression (5.12) of the constant $\alpha(v)$.

If there is a constant $\delta = \delta_0 + \delta_1 \mathbf{i}$ such that

$$\delta \cdot S(v) + T(v) = 0,$$

then $\delta = -t_{11} = t_{22}$. Now we prove that

$$-t_{11} \cdot s_{12} + t_{12} \neq 0.$$

In fact, a straightforward calculation shows

$$\begin{aligned} -t_{11} \cdot s_{12} + t_{12} &= -\frac{(1+|v|^2) \cdot v \cdot (1+\bar{v}^2)}{|v|^2(\bar{v}-v)} \cdot \frac{v(1-\bar{v}^2)}{2|v|^2} + 2 \frac{(1+|v|^2)(1+\bar{v}^2)}{(1-\bar{v}^2)(v-\bar{v})} \\ (5.17) \quad &= \frac{(1+|v|^2)(1+\bar{v}^2)^3}{2(v-\bar{v})\bar{v}^2(1-\bar{v}^2)} \neq 0, \end{aligned}$$

since we assumed that $v - \bar{v} \neq 0$ and $v^2 \neq -1$. □

(I-b) In this case we assume $v^2 = -1$. That is, there is a point $p = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ on the fiber of $q \in \Sigma_{GM}^7$ such that $v = xw^{-1} = \pm \mathbf{i}$. Then, according to the group action the point

$$p' = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} \cdot p \cdot \begin{pmatrix} -\mathbf{j} & 0 \\ 0 & 1 \end{pmatrix}$$

also lies in $\pi_{GM}^{-1}(q)$. Hence we may assume that $v = xw^{-1} = \mathbf{i}$. Then matrices $p \in \text{Sp}(2)$ in the fiber $(\pi_{GM})^{-1}(q)$ have the form

$$(5.18) \quad p = \begin{pmatrix} \mathbf{i}w & y \\ w & \mathbf{i}y \end{pmatrix}, \quad \forall w, \forall y \text{ with } |w|^2 = |y|^2 = \frac{1}{2}.$$

The space $Ad_p(\mathfrak{h}_p)$ is spanned by the 4 matrices

$$\begin{aligned} u_0 = u_0(\mathbf{i}) &= \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad u_{\mathbf{i}} = u_{\mathbf{i}}(\mathbf{i}) = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \\ u_{\mathbf{j}} = u_{\mathbf{j}}(\mathbf{i}) &= \begin{pmatrix} \mathbf{j} & \mathbf{k} \\ \mathbf{k} & -\mathbf{j} \end{pmatrix}, \quad u_{\mathbf{k}} = u_{\mathbf{k}}(\mathbf{i}) = \begin{pmatrix} \mathbf{k} & -\mathbf{j} \\ -\mathbf{j} & -\mathbf{k} \end{pmatrix}. \end{aligned}$$

Their commutators are given as

$$\begin{aligned} [u_0, u_{\mathbf{i}}] &= 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad [u_0, u_{\mathbf{j}}] = -2 \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}, \quad [u_0, u_{\mathbf{k}}] = -2 \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}, \\ [u_{\mathbf{i}}, u_{\mathbf{j}}] &= 2 \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix}, \quad [u_{\mathbf{i}}, u_{\mathbf{k}}] = -2 \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}, \quad [u_{\mathbf{j}}, u_{\mathbf{k}}] = 4 \begin{pmatrix} \mathbf{i} & 1 \\ -1 & \mathbf{i} \end{pmatrix}, \end{aligned}$$

and we have

$$\begin{aligned} (5.19) \quad Ad_{p^{-1}}([u_0, u_{\mathbf{i}}]) &= 4 \begin{pmatrix} -\overline{w}\mathbf{i}w & 0 \\ 0 & \overline{y}\mathbf{i}y \end{pmatrix}, \quad Ad_{p^{-1}}([u_0, u_{\mathbf{j}}]) = 4 \begin{pmatrix} 0 & \overline{w}\mathbf{k}y \\ \overline{y}\mathbf{k}w & 0 \end{pmatrix}, \\ Ad_{p^{-1}}([u_0, u_{\mathbf{k}}]) &= -4 \begin{pmatrix} 0 & \overline{w}\mathbf{j}y \\ \overline{y}\mathbf{j}w & 0 \end{pmatrix}, \quad Ad_{p^{-1}}([u_{\mathbf{j}}, u_{\mathbf{k}}]) = 16 \begin{pmatrix} 0 & 0 \\ 0 & \overline{y}\mathbf{i}y \end{pmatrix}. \end{aligned}$$

Moreover, note that

$$\begin{aligned} Ad_{p^{-1}}(u_0) &= 2 \begin{pmatrix} 0 & \overline{w}\mathbf{i}y \\ \overline{y}\mathbf{i}w & 0 \end{pmatrix}, \quad Ad_{p^{-1}}(u_{\mathbf{i}}) = 2 \begin{pmatrix} 0 & \overline{w}y \\ \overline{y}w & 0 \end{pmatrix} \\ Ad_{p^{-1}}(u_{\mathbf{j}}) &= 4 \begin{pmatrix} 0 & 0 \\ 0 & \overline{y}\mathbf{j}y \end{pmatrix}, \quad Ad_{p^{-1}}(u_{\mathbf{k}}) = 4 \begin{pmatrix} 0 & 0 \\ 0 & \overline{y}\mathbf{k}y \end{pmatrix} \end{aligned}$$

Therefore we obtain for all $\lambda = -\overline{\lambda} \in \mathbb{H}$:

$$\begin{aligned} (5.20) \quad \theta^{\lambda^+}(Ad_{p^{-1}}(u_0)) &= 0, \quad \theta^{\lambda^+}(Ad_{p^{-1}}(u_{\mathbf{i}})) = 0, \\ \theta^{\lambda^+}(Ad_{p^{-1}}(u_{\mathbf{j}})) &= 0, \quad \theta^{\lambda^+}(Ad_{p^{-1}}(u_{\mathbf{k}})) = 0, \\ \theta^{\lambda^+}(Ad_{p^{-1}}([u_0, u_{\mathbf{j}}])) &= 0, \quad \theta^{\lambda^+}(Ad_{p^{-1}}([u_0, u_{\mathbf{k}}])) = 0, \\ \theta^{\lambda^+}(Ad_{p^{-1}}([u_{\mathbf{j}}, u_{\mathbf{k}}])) &= 0. \end{aligned}$$

That is the $(1,1)$ -component of each among these 7 matrices is zero. Now, put $w = a_0 + a_1\mathbf{i} + b_0\mathbf{j} + b_1\mathbf{k} = a + b\mathbf{j} = -\mathbf{i}x \in \mathbb{H}$ and consider ℓ_ρ in (5.1),

Proposition 5.5. *Let $w = a_0 + a_1\mathbf{i} + b_0\mathbf{j} + b_1\mathbf{k} = a + b\mathbf{j} = -\mathbf{i}x \in \mathbb{H}$. In this case $|w|^2 = |x|^2 = |a|^2 + |b|^2 = \frac{1}{2}$. Then it follows with the above notation:*

(i) If $|a|^2 - |b|^2 \neq \frac{1}{4}$, then the 10 matrices

$$\begin{aligned}\ell_{\mathbf{i}} &= \begin{pmatrix} \mathbf{i} + \mathbf{i}w\mathbf{i}\overline{w} & -\mathbf{i}w\mathbf{i}\overline{w} \\ w\mathbf{i}\overline{w} & \mathbf{i} - w\mathbf{i}\overline{w} \end{pmatrix}, \quad \ell_{\mathbf{j}} = \begin{pmatrix} \mathbf{j} + \mathbf{i}w\mathbf{j}\overline{w} & -\mathbf{i}w\mathbf{j}\overline{w} \\ w\mathbf{j}\overline{w} & \mathbf{j} - w\mathbf{j}\overline{w} \end{pmatrix}, \\ \ell_{\mathbf{k}} &= \begin{pmatrix} \mathbf{k} + \mathbf{i}w\mathbf{k}\overline{w} & -\mathbf{i}w\mathbf{k}\overline{w} \\ w\mathbf{k}\overline{w} & \mathbf{k} - w\mathbf{k}\overline{w} \end{pmatrix}, \\ u_0 &= \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad u_{\mathbf{i}} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad u_{\mathbf{j}} = \begin{pmatrix} \mathbf{j} & \mathbf{k} \\ \mathbf{k} & -\mathbf{j} \end{pmatrix}, \quad u_{\mathbf{k}} = \begin{pmatrix} \mathbf{k} & -\mathbf{j} \\ -\mathbf{j} & -\mathbf{k} \end{pmatrix}, \\ F_{\mathbf{i}} &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}[u_0, u_{\mathbf{i}}], \quad F_{\mathbf{j}} := \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} = -\frac{1}{2}[u_0, u_{\mathbf{j}}], \\ F_{\mathbf{k}} &:= \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix} = -\frac{1}{2}[u_0, u_{\mathbf{k}}]\end{aligned}$$

are linearly independent over \mathbb{R} and span $\mathfrak{sp}(2)$.

(ii) If $|a|^2 - |b|^2 = \frac{1}{4}$, then the 10 matrices $\ell_{\mathbf{i}}, \ell_{\mathbf{j}}, \ell_{\mathbf{k}}, u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, F_{\mathbf{j}}, F_{\mathbf{k}}$ and

$$F'_{\mathbf{i}} := \begin{pmatrix} \mathbf{i} & 1 \\ -1 & \mathbf{i} \end{pmatrix} = \frac{1}{4}[u_{\mathbf{j}}, u_{\mathbf{k}}]$$

are linearly independent over \mathbb{R} and span $\mathfrak{sp}(2)$.

Proof. (i): Let $\mu_i, c_i, d_i \in \mathbb{R}$ and assume

$$(5.21) \quad U = \mu_1 \ell_{\mathbf{i}} + \mu_2 \ell_{\mathbf{j}} + \mu_3 \ell_{\mathbf{k}} + c_0 u_0 + c_1 u_{\mathbf{i}} + c_2 u_{\mathbf{j}} + c_3 u_{\mathbf{k}} \\ + d_1 F_{\mathbf{i}} + d_2 F_{\mathbf{j}} + d_3 F_{\mathbf{k}} = 0.$$

Then we conclude that $\theta^{\lambda^+}(Ad_{p^{-1}}(U)) = 0$ for any $\lambda = -\overline{\lambda} \in \mathbb{H}$. If we put $\mu = \mu_1 \mathbf{i} + \mu_2 \mathbf{j} + \mu_3 \mathbf{k}$, then the identities (5.20) together with

$$Ad_{p^{-1}}(F_{\mathbf{i}}) = 2 \begin{pmatrix} -\overline{w}\mathbf{i}w & 0 \\ 0 & \overline{y}\mathbf{i}y \end{pmatrix}$$

imply that

$$(5.22) \quad \overline{x}\mu x + \overline{w}\mu w - \mu + d_1(-\overline{w}x + \overline{x}w) = -\overline{w}\mathbf{i}\mu\mathbf{i}w + \overline{w}\mu w - \mu - 2d_1\overline{w}\mathbf{i}w = 0.$$

Hence

$$w\mu\overline{w} = \frac{\mu}{4} - \frac{\mathbf{i}\mu\mathbf{i}}{4} - \frac{d_1}{2}\mathbf{i}.$$

From this we have

$$(5.23) \quad x\mu\overline{x} = -\mathbf{i}w\mu\overline{w}\mathbf{i} = -\mathbf{i} \left(\frac{\mu}{4} - \frac{\mathbf{i}\mu\mathbf{i}}{4} - \frac{d_1}{2}\mathbf{i} \right) \mathbf{i} = \frac{\mu}{4} - \frac{\mathbf{i}\mu\mathbf{i}}{4} - red \frac{d_1}{2}\mathbf{i} = w\mu\overline{w}.$$

The equation (5.21) is rewritten as

$$(5.24) \quad \mu - x\mu\overline{x} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} + d_2 \mathbf{j} + d_3 \mathbf{k} = 0,$$

$$(5.25) \quad -x\mu\overline{w} + c_0 \mathbf{i} + c_2 \mathbf{k} - c_3 \mathbf{j} + d_1 = 0,$$

$$(5.26) \quad \mu - w\mu\overline{w} - c_1 \mathbf{i} - c_2 \mathbf{j} - c_3 \mathbf{k} + d_2 \mathbf{j} + d_3 \mathbf{k} = 0.$$

Using the identity (5.23) we obtain from (5.24) and (5.26):

$$c_1 = c_2 = c_3 = 0.$$

Then the equation (5.25) becomes

$$w\mu\overline{w} - c_0 + d_1 \mathbf{i} = 0,$$

which implies $c_0 = 0$ and

$$(5.27) \quad w\mu\bar{w} = -d_1\mathbf{i}.$$

Then the (5.26)

$$\mu - w\mu\bar{w} + d_2\mathbf{j} + d_3\mathbf{k} = \mu + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} = 0$$

gives $\mu_i = -d_i$ for $i = 1, 2, 3$ and (5.27) shows that

$$w\mu\bar{w} = \mu_1\mathbf{i}$$

This implies

$$\mu_1(|a|^2 - |b|^2) = \frac{\mu_1}{4}.$$

Hence $\mu_1 = 0$ and the identity $w\mu\bar{w} = 0$ implies that $\mu_2 = \mu_3 = 0$, which shows our assertion.

(ii): As in the case (i) we assume that

$$(5.28) \quad \mu_1\ell_{\mathbf{i}} + \mu_2\ell_{\mathbf{j}} + \mu_3\ell_{\mathbf{k}} + c_0u_0 + c_1u_{\mathbf{i}} + c_2u_{\mathbf{j}} + c_3u_{\mathbf{k}} + d_1F'_{\mathbf{i}} + d_2F_{\mathbf{j}} + d_3F_{\mathbf{k}} = 0.$$

By using the same arguments as before this equation is separated into a system of three equations:

$$(5.29) \quad \mu - x\mu\bar{x} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} = 0,$$

$$(5.30) \quad -x\mu\bar{w} + c_0\mathbf{i} + c_2\mathbf{k} - c_3\mathbf{j} + d_1 = 0,$$

$$(5.31) \quad \mu - w\mu\bar{w} - c_1\mathbf{i} - c_2\mathbf{j} - c_3\mathbf{k} + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} = 0.$$

Instead of the relation (5.22), we have

$$\bar{x}\mu x + \bar{w}\mu w - \mu = -\bar{w}\mathbf{i}\mu\bar{w} + \bar{w}\mu w - \mu = 0$$

since in this case the $(1, 1)$ component of $Ad_{p^{-1}}(F'_{\mathbf{i}})$ vanishes, cf. (5.19). However, one still has the equality (5.23):

$$x\mu\bar{x} = w\mu\bar{w}.$$

Hence $c_1 = c_2 = c_3 = 0$ and from (5.30) we see that $c_0 = 0$. Now (5.31) gives

$$\mu - w\mu\bar{w} + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} = \mu + 2d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} = 0$$

and therefore

$$\mu_1 = -2d_1.$$

Equation (5.30)

$$w\mu\bar{w} = \frac{1}{2}\mu_1\mathbf{i}$$

implies the equality

$$2\mu_1 = \mu_1(|a|^2 - |b|^2) = \frac{\mu_1}{4}.$$

Hence $\mu_1 = 0$, which also implies $\mu = 0$, since $|\mu| = |\mu_1|$. We have proved (ii). \square

Case (I-r): If $v_1 = 0$, i.e., when $v \neq 0$ is real, instead of $U_{\mathbf{j}}$ and $U_{\mathbf{k}}$, we choose the matrices $[u_0, u_{\mathbf{j}}]$ and $[u_0, u_{\mathbf{k}}]$.

Proposition 5.6. *With the notation in (5.1) the 10 matrices*

$$\left\{ \ell_{\mathbf{i}}, \ell_{\mathbf{j}}, \ell_{\mathbf{k}}, u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, [u_0, u_{\mathbf{i}}], [u_0, u_{\mathbf{j}}], [u_0, u_{\mathbf{k}}] \right\}$$

form a basis of $\mathfrak{sp}(2)$.

Proof. From the explicit list of commutators (5.11) we observe that

$$\mathrm{Tr}([u_0, u_\rho]) = \mathrm{Tr}(u_\rho) = 0, \quad \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

So the linear independence of the systems:

$$\mathcal{S}_1 := \{\ell_{\mathbf{i}}, \ell_{\mathbf{j}}, \ell_{\mathbf{k}}\} \quad \text{and} \quad \mathcal{S}_2 := \{u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, [u_0, u_{\mathbf{i}}], [u_0, u_{\mathbf{j}}], [u_0, u_{\mathbf{k}}]\}$$

is proved in the same way as in case (I-a). Linear independence of the seven matrices in \mathcal{S}_2 follows in the same way as the standard case in §3. \square

Case (II): Now we assume that $x = 0$ or $w = 0$. Then (5.5) shows that

$$(5.32) \quad \mathrm{Ad}_p(\mathfrak{h}_p) = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \mid b \in \mathbb{H} \right\}.$$

Choose the basis $u_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $u_{\mathbf{i}} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$, $u_{\mathbf{j}} = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}$, $u_{\mathbf{k}} = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix}$ of the space (5.32).

Proposition 5.7. *With the notation in (5.1) the 10 matrices*

$$\left\{ \ell_{\mathbf{i}}, \ell_{\mathbf{j}}, \ell_{\mathbf{k}}, u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}, [u_0, u_{\mathbf{i}}], [u_0, u_{\mathbf{j}}], [u_0, u_{\mathbf{k}}] \right\}$$

form a basis of $\mathfrak{sp}(2)$.

5.4. Final part of the proof. We complete the proof of the main Theorem 4.1 for the cases (I-a) and (I-b) based on Proposition 5.4 and Proposition 5.5, respectively. The remaining cases are proved in the same way via Propositions 5.6, 5.7, and 2.3.

Case (I-a): Let $q \in \Sigma_{GM}^7$ and take a point $p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \mathrm{Sp}(2)$ in the fiber such that

$$x \cdot w^{-1} = v_0 + v_1 \mathbf{i} := v \quad \text{and} \quad v_1 \neq 0, \quad v^2 \neq -1.$$

Locally around p we define vector fields \tilde{X}^ρ , $\rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ horizontal to the fibration $\mathrm{Sp}(2) \rightarrow \Sigma_{GM}^7$ and taking values in H^G such that

$$(dL_p)_{Id}(\mathrm{Ad}_{p^{-1}}(u_\rho)) = \tilde{X}_p^\rho \quad \text{for} \quad \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\},$$

where $u_0, u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}}$ are the matrices in $\mathrm{Ad}_p(\mathfrak{h}_p)$ defined in (5.7) - (5.10).

Proposition 5.8. *The local vector fields*

$$X^\rho := (d\pi_{GM})(\tilde{X}^\rho), \quad \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\},$$

take values in \mathcal{H}^Σ and the seven tangent vectors at $q = \pi_{GM}(p)$

$$\begin{aligned} & X_q^0, X_q^{\mathbf{i}}, X_q^{\mathbf{j}}, X_q^{\mathbf{k}}, \\ & [(d\pi_{GM})(\tilde{X}^0), (d\pi_{GM})(\tilde{X}^{\mathbf{i}})]_q = [X^0, X^{\mathbf{i}}]_q, \\ & [\alpha(v)(d\pi_{GM})_p(\tilde{X}^0), (d\pi_{GM})_p(\tilde{X}^{\mathbf{j}})]_q - [(d\pi_{GM})_p(\tilde{X}^{\mathbf{i}}), (d\pi_{GM})_p(\tilde{X}^{\mathbf{k}})]_q \\ & = \alpha(v)[X^0, X^{\mathbf{j}}]_q - [X^{\mathbf{i}}, X^{\mathbf{k}}]_q \\ & [\alpha(v)(d\pi_{GM})_p(\tilde{X}^0), (d\pi_{GM})_p(\tilde{X}^{\mathbf{k}})]_q + [(d\pi_{GM})_p(\tilde{X}^{\mathbf{i}}), (d\pi_{GM})_p(\tilde{X}^{\mathbf{j}})]_q \\ & = \alpha(v)[X^0, X^{\mathbf{k}}]_q + [X^{\mathbf{i}}, X^{\mathbf{j}}]_q \end{aligned}$$

span the tangent space $T_q(\Sigma_{GM}^7)$.

Proof. We define $Z_i \in \mathfrak{sp}(2)$ ($i = 1, 2, 3$) by

$$\begin{aligned} (dL_p)_{Id}(Z_1) &= [\tilde{X}^0, \tilde{X}^{\mathbf{i}}]_p, \\ (dL_p)_{Id}(Z_2) &= \alpha(v)[\tilde{X}^0, \tilde{X}^{\mathbf{j}}]_p - [\tilde{X}^{\mathbf{i}}, \tilde{X}^{\mathbf{k}}]_p, \\ (dL_p)_{Id}(Z_3) &= \alpha(v)[\tilde{X}^0, \tilde{X}^{\mathbf{k}}]_p + [\tilde{X}^{\mathbf{i}}, \tilde{X}^{\mathbf{j}}]_p, \end{aligned}$$

where $\alpha(v) \in \mathbb{H}$ is given in (5.12). Then, by Lemma 5.2 we have:

$$\begin{aligned} 0 &= \text{Tr}(Ad_p(Z_1) + [u_0, u_{\mathbf{i}}]) = \text{Tr}(Ad_p(Z_1)) = \text{Tr}(Ad_p(Z_1) - [u_0, u_{\mathbf{i}}]), \\ 0 &= \text{Tr}(Ad_p(Z_2) + \alpha(v)[u_0, u_{\mathbf{j}}] - [u_{\mathbf{i}}, u_{\mathbf{k}}]) = \text{Tr}(Ad_p(Z_2)) \\ &= \text{Tr}(Ad_p(Z_2) - \alpha(v)[u_0, u_{\mathbf{j}}] + [u_{\mathbf{i}}, u_{\mathbf{k}}]), \\ 0 &= \text{Tr}(Ad_p(Z_3) + \alpha(v)[u_0, u_{\mathbf{k}}] + [u_{\mathbf{i}}, u_{\mathbf{j}}]) = \text{Tr}(Ad_p(Z_3)) \\ &= \text{Tr}(Ad_p(Z_3) - \alpha(v)[u_0, u_{\mathbf{k}}] - [u_{\mathbf{i}}, u_{\mathbf{j}}]). \end{aligned}$$

Also Lemma 5.1 implies that

$$\begin{aligned} < \lambda^+, Z_1 - Ad_{p^{-1}}([u_0, u_{\mathbf{i}}]) > = 0, \\ < \lambda^+, Z_2 - Ad_{p^{-1}}(\alpha(v)[u_0, u_{\mathbf{j}}] - [u_{\mathbf{i}}, u_{\mathbf{k}}]) > = 0, \\ < \lambda^+, Z_3 - Ad_{p^{-1}}(\alpha(v)[u_0, u_{\mathbf{k}}] + [u_{\mathbf{i}}, u_{\mathbf{j}}]) > = 0, \end{aligned}$$

for any $\lambda = -\bar{\lambda} \in \mathbb{H}$. Therefore, the vectors

$$\begin{aligned} &Ad_p(Z_1) - [u_0, u_{\mathbf{i}}], \quad Ad_p(Z_2) - \alpha(v)[u_0, u_{\mathbf{j}}] + [u_{\mathbf{i}}, u_{\mathbf{k}}], \\ &Ad_p(Z_3) - \alpha(v)[u_0, u_{\mathbf{k}}] - [u_{\mathbf{i}}, u_{\mathbf{j}}] \end{aligned}$$

belong to $Ad_p(\mathfrak{h}_p)$. Note that in Proposition 5.4 we may replace $[u_0, u_{\mathbf{i}}]$, $U_{\mathbf{j}}$ and $U_{\mathbf{k}}$ by $Ad_p(Z_1)$, $Ad_p(Z_2)$ and $Ad_p(Z_3)$, respectively. This proves Case (I-a). \square

Case (I-b): We treat case (i) of Proposition 5.5 and remark that (ii) is proved in the same way as the case (I-a). Let $q \in \Sigma_{GM}^7$ and fix a point

$$p = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \text{Sp}(2)$$

in the fiber such that $v = x \cdot w^{-1} = \mathbf{i}$. Locally around p we define vector fields \tilde{Y}^ρ , $\rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ horizontal to the fibration $\text{Sp}(2) \rightarrow \Sigma_{GM}^7$ and taking values in H^G such that

$$(dL_p)_{Id}(Ad_{p^{-1}}(u_\rho(\mathbf{i}))) = \tilde{Y}_p^\rho \quad \text{for } \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

Here $u_0(\mathbf{i}) = u_0$, $u_{\mathbf{i}}(\mathbf{i}) = u_{\mathbf{i}}$, $u_{\mathbf{j}}(\mathbf{i}) = u_{\mathbf{j}}$, $u_{\mathbf{k}}(\mathbf{i}) = u_{\mathbf{k}}$ are the matrices in $Ad_p(\mathfrak{h}_p)$ defined in (5.7) - (5.10) with $v = \mathbf{i}$. Let

$$Y^\rho := (d\pi_{GM})(\tilde{Y}^\rho) \quad \text{with } \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

be the horizontal vector fields around $q = \pi_{GM}(p)$ taking values in \mathcal{H}^Σ .

Proposition 5.9. *The seven tangent vectors at $q = \pi_{GM}(p)$*

$$\begin{aligned} &Y_q^0, Y_q^{\mathbf{i}}, Y_q^{\mathbf{j}}, Y_q^{\mathbf{k}}, \\ &[(d\pi_{GM})(\tilde{Y}^0), (d\pi_{GM})(\tilde{Y}^{\mathbf{i}})]_q = [Y^0, Y^{\mathbf{i}}]_q, \\ &[(d\pi_{GM})(\tilde{Y}^0), (d\pi_{GM})(\tilde{Y}^{\mathbf{j}})]_q = [Y^0, Y^{\mathbf{j}}]_q, \\ &[(d\pi_{GM})(\tilde{Y}^0), (d\pi_{GM})(\tilde{Y}^{\mathbf{k}})]_q = [Y^0, Y^{\mathbf{k}}]_q \end{aligned}$$

span the tangent space $T_q(\Sigma_{GM}^7)$.

Proof. We define $W_i \in \mathfrak{sp}(2)$ ($i = 1, 2, 3$) through the relations

$$\begin{aligned}(dL_p)_{Id}(W_1) &= [\tilde{Y}^0, \tilde{Y}^i]_p, \\ (dL_p)_{Id}(W_2) &= [\tilde{Y}^0, \tilde{Y}^j]_p, \\ (dL_p)_{Id}(W_3) &= [\tilde{Y}^0, \tilde{Y}^k]_p,\end{aligned}$$

By the Lemmas 5.2 and 5.1 together with the explicit expressions of the commutators

$$[u_0(\mathbf{i}), u_{\mathbf{i}}(\mathbf{i})] = [u_0, u], \quad [u_0(\mathbf{i}), u_{\mathbf{j}}(\mathbf{i})] = [u_0, u_{\mathbf{k}}], \quad \text{and} \quad [u_0(\mathbf{i}), u_{\mathbf{k}}(\mathbf{i})] = [u_0, u_{\mathbf{k}}]$$

in (5.11) and the identities in (5.19) we obtain:

$$\begin{aligned}0 &= \text{Tr}(Ad_p(W_1) + [u_0, u_{\mathbf{i}}]) \\ &= \text{Tr}(Ad_p(W_1)) = \text{Tr}(Ad_p(W_1) - [u_0, u_{\mathbf{i}}]), \\ 0 &= \langle \lambda^+, W_1 - Ad_{p^{-1}}([u_0, u_{\mathbf{i}}]) \rangle, \\ 0 &= \text{Tr}(Ad_p(W_2) + [u_0, u_{\mathbf{j}}]) = \text{Tr}(Ad_p(W_2) - (-[u_0, u_{\mathbf{j}}])), \\ 0 &= \langle \lambda^+, W_2 - Ad_{p^{-1}}([u_0, u_{\mathbf{j}}]) \rangle = \langle \lambda^+, W_2 \rangle \\ &= \langle \lambda^+, W_2 - (Ad_{p^{-1}}(-[u_0, u_{\mathbf{j}}])) \rangle, \\ 0 &= \text{Tr}(Ad_p(W_3) + [u_0, u_{\mathbf{k}}]) = \text{Tr}(Ad_p(W_3) - (-[u_0, u_{\mathbf{k}}])), \\ 0 &= \langle \lambda^+, W_3 - Ad_{p^{-1}}([u_0, u_{\mathbf{k}}]) \rangle = \langle \lambda^+, W_3 \rangle \\ &= \langle \lambda^+, W_3 - (Ad_{p^{-1}}(-[u_0, u_{\mathbf{k}}])) \rangle.\end{aligned}$$

Hence we obtain

$$Ad_p(W_1) - [u_0, u_{\mathbf{i}}], Ad_p(W_2) + [u_0, u_{\mathbf{j}}], Ad_p(W_3) + [u_0, u_{\mathbf{k}}] \in Ad_p(\mathfrak{h}_{\mathbf{p}}),$$

Analogously to the last case (I-a) this implies that $\{Y^\rho \mid \rho = 0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ together with

$$(d\pi_{GM})_p \circ (dL_p)_{Id}(W_i) = (d\pi_{GM})_p([\tilde{Y}^0, \tilde{Y}^\rho]) = [Y^0, Y^\rho]_q, \rho \in \{0, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

span the tangent space $T_q(\Sigma_{GM}^7)$. \square

It would be interesting to decide whether some or all of the remaining 26 exotic 7 spheres admit a higher co-dimensional sub-Riemannian structure. According to [5, 11, 14] the Gromoll-Meyer exotic sphere Σ_{GM}^7 is the only exotic sphere that is modeled by a biquotient of a compact group. If an exotic sphere is realized as a base space of a principal bundle in which the total space is not a group the method of the present paper are not applicable and new strategies are required for attacking this question.

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